Stability of Quantum Isolated Horizon with An Energy Spectrum Linear in Area

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Considering the microstates of the SU(2) Chern-Simons theory of a generic Quantum Isolated Horizon(QIH), the *microcanonical* entropy is derived in a *model independent* method. It is shown that a QIH, with *energy spectrum linear in area* ($E = A/8\pi\ell_p$), is locally *unstable* as a thermodynamic system. The result is derived in two different ways. Firstly, the specific heat of the QIH is shown to be negative definite through a quantum statistical analysis. Then, it is shown, in the thermal holographic approach, that the canonical partition function of the QIH diverges under Gaussian thermal fluctuations of such energy spectrum, implying local instability of such a QIH as a thermodynamic system.

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I. INTRODUCTION

The effective description of a Quantum Isolated Horizon (QIH) is given by a three dimensional $SU(2)$ Chern Simons (CS) theory coupled to point like sources on punctures made by the edges of spin network describing the bulk quantum geometry [1–3]. The punctures are associated with $SU(2)$ spins and designated by quantum numbers j, m. The microstates of the CS theory give rise to the entropy of the QIH. Detailed quantum statistical formulation of the QIH using these microstates lead to relevant thermodynamic properties of the QIH.

The CS theory has an $SU(2)$ gauge invariance which can be partially fixed to a residual $U(1)$ gauge group. Microstates of the horizon have been counted both by using $SU(2)$ and $U(1)$ CS theory in the literature. However, it has been shown [4] that the entropy of a QIH is independent of this gauge redundancy, as it should be, entropy being a physical quantity. Recently, it has been claimed in [5], considering $U(1)$ microstates, that an uncharged, non-rotating QIH, having an energy spectrum *linear in area* is locally stable as a thermodynamic system. In that paper, it is also postulated that the area operator and an operator giving the number of punctures on the QIH be treated as *independent* observables in that their *fluctuations* be considered as independent of each other. One notes, though, that the linearity of the mass spectrum as a function of the area of the QIH is only a *model* for the QIH, one which relates the equilibrium values of the area and the number-of-punctures operators as an 'equation of state'. However, as we demonstrate explicitly below, this (or any other model expressed in terms of a mass spectrum of a QIH as a function of its area) seems not essential for derivation of the microcanonical entropy of a QIH as a function of the (arbitrarily specifiable, but macroscopically large) equilibrium values of the horizon area and number-of-punctures. Furthermore, we present a detailed stability analysis of a QIH having such an energy spectrum, but considering the $SU(2)$ microstates, to argue that such a spectrum in fact leads to *instabilities*. The stability analysis is carried out using two methods – one that is similar to the one followed in [5] and the other which is a completely independent approach. Both methods lead to the same conclusion which is another key result of this paper: *An uncharged, non-rotating QIH, having an energy spectrum linear in area is locally unstable as a thermodynamic system.* An outline of the contents of this paper is briefly described as follows.

Section (II) deals with calculation of the microcanonical entropy of the QIH from the microstates of $SU(2)$ CS theory. In section (III), the energy spectrum *assumed* for the QIH in [5] is rederived in a different approach to overcome a few difficulties in the original paper [5]. In section (IV) the thermodynamic stability of the QIH is analyzed in complete detail. Effects of both *quantum* and *thermal* fluctuations are incorporated. First, in subsection (IV A) thermodynamic quantities associated with the QIH are calculated using an explicit quantum statistical formulation. The specific heat is found to be negative definite showing the QIH to be locally *unstable* as a thermodynamic system. In subsection (IV B), the effect of thermal fluctuations on the

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stability of the QIH is investigated in the thermal holographic approach [11, 17], once again resulting in the local instability of the QIH. Finally, we conclude with a discussion in section (V) which includes an explanation of the reason why the claim about the stability of QIH made in [5] may not be valid.

II. $SU(2)$ MICROSTATES AND MICROCANONICAL ENTROPY

The counting of the $SU(2)$ microstates have been done earlier in [13] using the relation between the Hilbert space of the $SU(2)$ CS theory on the boundary(QIH) with the space of conformal blocks of the Wess-Zumino model on the boundary 2-sphere. The number of microstates for a spin configuration $\{j_1, j_2, ..., j_N\}$ is given by

$$d[\{j_l\}] = \frac{2}{k+2} \sum_{q=0}^{k/2} \frac{\prod_{l=1}^{N} \sin\left[\frac{(2j_l+1)(2q+1)\pi}{k+2}\right]}{\left\{\sin\left[\frac{(2q+1)\pi}{k+2}\right]\right\}^{N-2}} \tag{II.1}$$

where k is the level of the $SU(2)$ CS theory and N is the total number of punctures. Alternatively, such a spin configuration can be represented by a set $\{s_j\}$ where s_j is the number of punctures with spin value j.[1] The number of microstates for a particular spin configuration $\{s_j\}$ in the $SU(2)$ approach is given by

$$d[\{s_j\}] = \frac{2}{k+2} \frac{\left(\sum_j s_j\right)!}{\prod_j s_j!} \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_j \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{s_j} \tag{II.2}$$

where $a = 2q+1$. The combinatorial factor arises from the choice of punctures [16]. Now, the spin configuration $\{s_j\}$ must obey the following constraints

$$\mathcal{C}_1 : \sum s_j = N \qquad \text{and} \qquad \mathcal{C}_2 : \sum_j s_j \sqrt{j(j+1)} = \frac{A}{8\pi\gamma\ell_p^2} \tag{II.3}$$

where $A =$ the area of the QIH , $N =$ total number of punctures of the QIH. Since we consider macroscopic black holes i.e. large QIH, arbitrary small fluctuations $\delta A(\ll A)$ and $\delta N(\ll N)$. Variation of $\log d[\{s_j\}]$ with respect to s_j, subject to the constraints $\mathcal{C}_1$ and $\mathcal{C}_2$, yields the dominant configuration which maximizes the entropy of the QIH. The variational equation is given by

$$\delta \log d[\{s_j\}] - \frac{\lambda}{8\pi\gamma\ell_p^2}\delta A - \sigma\delta N = 0 \tag{II.4}$$

where δ represents variation with respect to s_j, λ and σ are the Lagrange multipliers for $\mathcal{C}_1$ and $\mathcal{C}_2$ respectively. This yields the dominant configuration given by

$$\bar{s}_j = N M_j(k) e^{-\lambda\sqrt{j(j+1)}-\sigma} \tag{II.5}$$

where

$$\begin{aligned} M_j(k) &= \prod_{a=1}^{k+1} \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{\frac{f_a(k)}{f(k)}} \\ f(k) &= \sum_{a=1}^{k+1} f_a(k) = \sum_{a=1}^{k+1} \sin^2 \frac{a\pi}{k+2} \prod_j \left\{ \frac{\sin \frac{a\pi(2j+1)}{k+2}}{\sin \frac{a\pi}{k+2}} \right\}^{\bar{s}_j} \end{aligned}$$

Now, from (II.5) we have

$$\log \bar{s}_j = \log N + \log M_j(k) - \lambda\sqrt{j(j+1)} - \sigma$$

[1] $s_{1/2} =$ number of punctures with spin 1/2, $s_1 =$ number of punctures with spin 1 and so on.

Equation (II.2) can be rewritten as

$$d[\{s_j\}] = \frac{2}{k+2}\,\frac{N!}{\prod_j s_j!}\,f(k) \tag{II.6}$$

Hence, from (II.6), for dominant configuration $\bar{s}_j$, we have

$$\log d[\{\bar{s}_j\}] = \log\frac{2}{k+2} + \log N! - \sum_j \log(\bar{s}_j!) + \log f(k)$$

Since we are investigating macroscopic black holes, we consider large number of intersections of the bulk spin networks with the horizon for each spin value. Henceforth, we take the limit $s_j \to \infty$ for all the calculations. So, one can calculate the microcanonical entropy(S_{MC}) for the dominant configuration in the appropriate limit and show that

$$\begin{aligned} S_{MC} &= \lim_{\bar{s}_j\to\infty} \log d[\{\bar{s}_j\}] \\ &= \lambda\frac{A}{8\pi\gamma\ell_p^2} + \log\frac{2f(k)}{(k+2)\prod_j \{M_j(k)\}^{\bar{s}_j}} + N\sigma(\lambda) \end{aligned}$$

using Stirling's approximation, $\mathcal{C}_1$ and $\mathcal{C}_2$. The logarithmic term appearing in the above expression of entropy is a physically relevant term. To observe the significance of that logarithm, one should take the limit $k \to \infty$ which is a natural limit for macroscopic black holes with large area ($A \gg \ell_p^2$) since $A \propto k$ [1, 9]. It has been shown in various ways in literature [14–16] that the above logarithmic term, in the limits $\bar{s}_j, k \to \infty$, gives the well known log correction to the entropy, that is

$$\lim_{k\to\infty} \log\frac{2f(k)}{(k+2)\prod_j \{M_j(k)\}^{s_j}} = -\frac{3}{2}\log A$$

where some constants are neglected. Therefore, using all the appropriate limits ($\bar{s}_j, k \to \infty$) for macroscopic black holes one can show that

$$S_{MC} = \frac{\lambda A}{8\pi\gamma\ell_p^2} + N\sigma - \frac{3}{2}\log\frac{A}{\ell_p^2} \tag{II.7}$$

where one can show by using (II.5) in $\mathcal{C}_1$ and $\mathcal{C}_2$ that the two Lagrange multipliers satisfy the following two equations

$$e^{\sigma} = \sum_j M_j(k) e^{-\lambda\sqrt{j(j+1)}} \tag{II.8}$$

$$\frac{\mathscr{A}}{N} = \frac{\sum_j M_j(k)\sqrt{j(j+1)}e^{-\lambda\sqrt{j(j+1)}}}{e^{\sigma}} \tag{II.9}$$

where $\mathscr{A} = A/8\pi\gamma\ell_p^2$. Now, the whole problem boils down to the task of solving λ and σ in terms of $\mathscr{A}/N$. Eliminating σ from the equations (II.8) and (II.9) one obtains

$$\frac{\mathscr{A}}{N} = \frac{\sum_j M_j(k)\sqrt{j(j+1)}e^{-\lambda\sqrt{j(j+1)}}}{\sum_j M_j(k)e^{-\lambda\sqrt{j(j+1)}}} \equiv F(\lambda) \tag{II.10}$$

In principle, given $\mathscr{A}/N$, one can solve (II.10) for $\lambda = \lambda(\mathscr{A}/N)$ by writing $\lambda = F^{-1}(\mathscr{A}/N)$ (as a functional inverse) and then evaluate σ from equation (II.8) explicitly as a function of $\mathscr{A}/N$. But, in general, this F^{-1} may not be unique i.e. λ can have *multiple* solutions $\lambda(\mathscr{A}/N)$. This possible ambiguity in the solution for λ may be parametrized in terms of a real constant λ_0 so that one may write $\lambda = F^{-1}(\mathscr{A}/N, \lambda_0)$. It is important to realize though that λ_0 most likely is *not* an arbitrary real parameter, but takes only a finite set of values consistent with the solutions of (II.10).

Further, the allowed values of λ_0 corresponding to multiple solutions for $\lambda = \lambda(\mathscr{A}/N)$ may be identified with the Barbero-Immirzi (BI) parameter (γ) (modulo a factor of 2π) if the following alternative method is adopted. Using (II.8) and (II.9) one can show that $\mathscr{A}/N = -d\sigma(\lambda)/d\lambda$, which on integration yields

$$\sigma(\lambda) = \frac{A}{8\pi\gamma\ell_p^2 N}(\lambda_0 - \lambda) + \sigma(\lambda_0) \tag{II.11}$$

Using (II.11) in (II.7), we finally obtain

$$S_{MC} = \frac{\lambda_0 A}{8\pi\gamma\ell_p^2} + N\sigma(\lambda_0) - \frac{3}{2}\log\frac{A}{\ell_p^2} \tag{II.12}$$

Reparametrizing $\lambda_0 = 2\pi\gamma$ now leads to the leading semiclassical Bekenstein-Hawking term linear $(A/4\ell_p^2)$, together with *quantum corrections*, for the *microcanonical* entropy of the QIH

$$S_{MC} = \frac{A}{4\ell_p^2} + N\sigma(\gamma) - \frac{3}{2}\log\frac{A}{\ell_p^2} \tag{II.13}$$

where $\sigma(\gamma) = \log\sum_j M_j(k)e^{-2\pi\gamma\sqrt{j(j+1)}}$. For convenience, from hereon, we choose $\ell_p = 1$. This reparametrization of the λ_0 in terms of the BI parameter implies that (II.13), as a formula for the *microcanonical* entropy for a QIH, holds *only* for a finite (or at best discrete) set of real positive values of the BI parameter, to be determined from the explicit solutions of the equations (II.8) and (II.9) i.e. *in principle, the allowed values of γ are determined.* This is somewhat different from the formula for the microcanonical entropy of a QIH derived in [5] where the BI parameter is left as an arbitrary real positive number. Of course it is only a technical matter to solve the equations (II.8) and (II.9) explicitly so as to determine the exactly allowed value(s) of γ; we hope to present this elsewhere [8].

III. ENERGY SPECTRUM OF QIH

An energy spectrum *linear in area*, for the QIH, has been *proposed in* [5] as a model of a black hole. In this section we focus on a few issues related to this model within classical general relativity and LQG.

The definition of 'energy' used in [5], addressed as 'Komar mass integral', is given by

$$E_r = -\frac{1}{8\pi}\int_{S_r} \nabla^a u^b dS_{ab} \tag{III.1}$$

where $u = \frac{\xi}{\sqrt{|\xi.\xi|}}$, ξ being a Killing vector, satisfies $\nabla_a\xi_b + \nabla_b\xi_a = 0$. S_r is a 2-sphere of radius r. For a Schwarzschild black hole, it is straightforward to show that $E_r = \frac{M}{2}\left(1 - \frac{2M}{r}\right)^{-\frac{1}{2}}$. The near horizon limit at $r = 2M + \epsilon$ is obtained to be $E \equiv E_{2M+\epsilon} \approx \frac{M}{2}\left(\frac{2M}{\epsilon}\right)^{\frac{1}{2}} = \frac{A}{8\pi\ell}$ (since $\epsilon \ll 2M$) using $\ell = 2(2M\epsilon)^{\frac{1}{2}}$ and $A = 16\pi M^2$. Following this approximation, the spectrum of the Hamiltonian operator has been proposed (in terms of the area spectrum in LQG [18, 19]) to be

$$\widehat{H}|j_1, j_2\cdots\rangle = \frac{\gamma\ell_p^2}{\ell}\sum_l \sqrt{j_l(j_l+1)})\ |j_1, j_2\cdots\rangle \tag{III.2}$$

where j_l taking values from the set $\{1/2, 1, 3/2, ...\}$ is the spin associated with the l-th puncture. γ is the Immirzi parameter and ℓ_p is the Planck length.

There are a few issues regarding the above derivation of energy spectrum in [5] which we draw the attention of the reader to :-

1. Clearly, $\lim_{r\to\infty} E_r = \frac{M}{2} \neq M$, the ADM mass of the Schwarzschild black hole. Also, it is not the Komar mass[2]. Hence, E_r is not the energy of the black hole. Most crucially, it is straightforward to show that the quantity E_r is an *unconserved* quantity. Hence, E_r can not be considered as a *physical* energy associated with a horizon.

2. A new length scale $\ell = 2(2M\epsilon)^{1/2}$ has had to be introduced to obtain $E \propto A$, where M is the ADM mass associated with the black hole spacetime. This, of course, varies for various black holes. ϵ is an arbitrary length scale satisfying $\epsilon \ll 2M$. Since there is no other physical input for ϵ in ref. [5], it can be considered as a tuning parameter which should be tuned *by hand* for different M's. So the physicality of this parameter, in any fundamental sense, is uncertain in this treatment.

[2] The Komar mass is obtained by replacing u by ξ in (III.1).

3. Last but not the least, the surface $r = 2M + \epsilon$ is not at all a null surface, rather a time-like one. It is a quasi-local time-like boundary of spacetime which can only be regarded as a sort of a 'stretched horizon' and NOT an isolated horizon (null inner boundary of spacetime). Strictly speaking, then, the area of this surface is not a Dirac observable in LQG [18], unlike the area operator corresponding to a QIH. Moreover, the quantum energy spectrum used in [5] is actually that of a stretched horizon and not applicable for a QIH unless it is simply *assumed* [7].

In what follows, we shall show that the above issues point to further conundra associated with the model of a black hole proposed in [5].

The Hamiltonian formulation of the classical phase space of spacetimes admitting internal boundaries (classical isolated horizons(CIH) at equilibrium) shows that there exists an energy associated with each CIH satisfying a first law[6]. A correct quantization of such a theory must lead to a horizon energy spectrum expressed in terms of the spectra of the operators corresponding to the other extensive variables of the first law (namely area, charge, angular momentum, etc.). Unfortunately, such things have not been done up till now. On the other hand, in quantum geometry[1], the full Hilbert space of a quantum black hole can be written as $\mathcal{H} = \mathcal{H}_{\mathcal{V}} \otimes \mathcal{H}_{\mathcal{S}}$ modulo some constraints, where $\mathcal{V}(\mathcal{S})$ stands for volume (surface). Thus, any generic state $|\Psi\rangle$, of the quantum black hole can be written as $|\Psi\rangle = |\Psi_{\mathcal{V}}\rangle \otimes |\Psi_{\mathcal{S}}\rangle$. Hence, any operator which acts on the states of the Hilbert space $\mathcal{H}$, say the Hamiltonian $\widehat{H}$, must have a form [11] $\widehat{H} = (\widehat{H}_{\mathcal{V}} \otimes \widehat{I}_{\mathcal{S}} + \widehat{I}_{\mathcal{V}} \otimes \widehat{H}_{\mathcal{S}})$ where $\widehat{I}$ represents identity operator. But the spectrum of this Hamiltonian is unknown. So, clearly there is a missing link between the classical and quantum theories of IH as far as the energy spectrum is concerned.

To avoid this missing link and proceed further, we *assume the spectrum of the Hamiltonian operator associated with the QIH to be linear in area*. The assumption is based on an analogy with the near horizon approximation of the classical ADM energy of a Schwarschild black hole which is given by $E_{ADM} = \lim_{r\to\infty} E_r = \lim_{r\to\infty} \frac{M}{1-\frac{2M}{r}} = M$ where r is the usual radial Schwarzschild coordinate. The near horizon limit of E_r at $r = 2M + \epsilon$ gives us $E_{2M+\epsilon} = M/(1 - \frac{2M}{2M+\epsilon}) \approx 2M^2/\epsilon = A/8\pi\epsilon$, since $\epsilon \ll 2M$ and $A = 16\pi M^2$. Making an analogy with this classical approximation, *we assume the energy spectrum of the QIH to be*[3]

$$E = \frac{A}{8\pi\ell_p} \tag{III.3}$$

Now, using quantum geometry one can show[11, 17] that the partition function of a quantum black hole is completely determined by surface states i.e. $Z = Z_{\mathcal{S}} = Tr_{\mathcal{S}} \exp -\beta\widehat{H}_{\mathcal{S}}$. This is nothing but the partition function for the QIH. Since, both the operators $\widehat{H}_{\mathcal{S}}$ and $\widehat{A}$ act on $|\Psi_{\mathcal{S}}\rangle$, following energy spectrum (III.3) the action of $\widehat{H}_{\mathcal{S}}$ on $|\Psi_{\mathcal{S}}\rangle$ can be written as

$$\widehat{H}_{\mathcal{S}}|\Psi_{\mathcal{S}}\rangle = \frac{1}{8\pi\ell_p}\widehat{A}|\Psi_{\mathcal{S}}\rangle$$

where $\widehat{A}$ is the area operator in LQG [18, 19]. So, the spectrum of $\widehat{H}_{\mathcal{S}}$ can be written in terms of the spectrum of $\widehat{A}$ as

$$\widehat{H}_{\mathcal{S}}|j_1, j_2\cdots\rangle = \left(\gamma\ell_p \sum_{l=1}^{N} \sqrt{j_l(j_l+1)}\right)|j_1, j_2\cdots\rangle$$

where γ is the Immirzi parameter and ℓ_p is the Planck length. j_l is the spin associated with the l th puncture, N being the total number of punctures. $|\Psi_{\mathcal{S}}\rangle \equiv |j_1, j_2....\rangle$ is a microstate of the $SU(2)$ CS theory on the QIH, having spin configuration $\{j_1, j_2, ..., j_N\}$. Thus, we have obtained an energy spectrum for uncharged, non-rotating QIH representing a quantum black hole.

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[3] We emphasize that unlike [5], the energy spectrum of QIH is *assumed* here, rather than being *derived*. The classical approximation only provides a ground for this assumption and nothing more than that.

IV. THERMODYNAMICS OF QIH

A. Explicit Quantum Statistical Stability Analysis

To get an insight of the thermodynamic properties of the QIH, we explore the canonical ensemble scenario where N is kept *fixed* and E is allowed to fluctuate. The prime task is to write down the canonical partition function for a spin configuration $\{s_j\}$ which is written as

$$Z(\beta, N) = \sum_{\{s_j\}} d[\{s_j\}] e^{-\beta E_{\{s_j\}}} \approx d[\{\bar{s}_j\}] e^{-\beta E} \tag{IV.1}$$

where E_j = energy associated with a spin j, $\sum_j \bar{s}_j E_j = E$ = energy of the QIH for $\{\bar{s}_j\}$(thermal equilibrium). The contributions from the sub-dominant configurations are neglected. β is the inverse temperature of the QIH given by $\beta = \partial S_{MC}/\partial E|_N$ which results in

$$\beta = 2\pi \left(1 - \frac{6}{A}\right) \tag{IV.2}$$

To calculate relevant thermodynamic quantities one needs to calculate the logarithm of the partition function in the appropriate limits ($\bar{s}_j, k \to \infty$). A straightforward calculation using equations (III.3), (IV.1) and (IV.2) yields

$$\log Z = N\sigma(\gamma) - \frac{3}{2}\log A + \frac{3}{2} \tag{IV.3}$$

The *average energy* of the QIH in the canonical ensemble can be calculated from (IV.3) using the usual thermodynamical relation $\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z$. Using $d\beta/dA = 12\pi/A^2$ and $E = A/8\pi$ it is straightforward to show that the average energy of the QIH is equal to its equilibrium energy i.e. $\langle E \rangle = E$. Following this, the *specific heat* of the QIH can be calculated using the usual thermodynamic formula $C = -\beta^2 \partial \langle E \rangle / \partial \beta$. A few steps of algebra lead to

$$C = -\frac{\beta^2 A^2}{96\pi^2}$$

where one has to use $\langle E \rangle = E = A/8\pi$ and $d\beta/dA = 12\pi/A^2$. The specific heat being *negative definite* one can conclude that an uncharged, non-rotating Quantum Isolated Horizon, having energy spectrum as (III.3), is locally *unstable* as a thermodynamic system.

The validity of *the first law* can be checked using (II.7) and (IV.4) and one can easily show that $dE = TdS$ is indeed satisfied.

From (IV.2) one can find the local *horizon temperature* to be

$$T = \frac{1}{2\pi}\left(1 + \frac{6}{A} + ...\right) \qquad [k_B = 1] \tag{IV.4}$$

This local horizon temperature contains a series which is identical to the correction terms obtained for the horizon temperature in [17] considering Gaussian thermal fluctuations about the equilibrium. The connection between these two may be a future issue of interest.

NOTE : Let us have a closer look at the canonical partition function. The *exact* canonical partition function, without any approximation, can be written as

$$Z(\beta, N) = \bar{Z}(\beta, N) + \delta(\beta, N)$$

where $\delta(\beta, N)$ is the contribution from *thermal fluctuations* (sub-dominant configurations $\{s_j\}$s other than $\{\bar{s}_j\}$) about equilibrium value ($\bar{Z}$) of the canonical partition function coming from the dominant configuration $\{\bar{s}_j\}$ whose spin distribution is given by eq. (II.5). Truly speaking, eq. (IV.1) is only $\bar{Z}$ and *not* Z. The effect of the thermal fluctuations is completely neglected ($\delta = 0$) in eq.(IV.1). This has a profound implication.

The canonical entropy is given by $S_C = \log Z + \beta \langle E \rangle$, which can be recast as

$$S_C \;=\; \bar{S}_C + \log(1 + \delta/\bar{Z})$$

where $\bar{S}_C = \log \bar{Z} + \beta E$ and $\langle E \rangle = E$ is used. If one calculates $\log \bar{Z} + \beta E$, a few steps of algebra leads to $\bar{S}_C = S_{MC}$. Therefore,

$$S_C = S_{MC} + \log(1 + \delta/\bar{Z})$$

If we do not take the effects of thermal fluctuations in canonical ensemble i.e. $\delta = 0$, then it is obvious that $S_C = S_{MC}$ at *all* temperatures. This is a very general result concerning a thermodynamic system [10] which also applies for black holes as has been shown earlier in the literature [12]. In fact this extra contribution from thermal fluctuations plays a very important role in analyzing the thermodynamic stability of the QIH [11] which we will discuss briefly in the next subsection. Sitting at the equilibrium and ignoring the thermal fluctuations lead to a physically incomplete scenario which apparently looks to give us an *ensemble independent* result [5]. Hence, to get the complete picture, we must take into account quantum and thermal fluctuations both.

B. Thermal Fluctuations in Canonical Ensemble

Starting from the canonical partition function of the QIH and considering Gaussian thermal fluctuations about an equilibrium configuration (saddle point)[10], one can actually derive an inequality between the energy and the microcanonical entropy of the QIH at equilibrium using appropriate units [11]. This inequality serves as the stability criterion for the QIH. It is as simple as

$$E > S_{MC} \tag{IV.5}$$

where E and S_{MC} are the energy and the microcanonical entropy of the QIH at equilibrium respectively. Since, here we know both the energy spectrum and the microcanonical entropy of the QIH, it is straightforward to compare (III.3) and (II.7) and see that the stability criterion (IV.5) is violated i.e.

$$\frac{A}{8\pi} < \frac{A}{4} - \frac{3}{2}\log A \Rightarrow E < S_{MC} \tag{IV.6}$$

Hence a QIH having an energy spectrum given by (III.3) is locally *unstable* as a thermodynamic system.

To be more explicit, if one calculates the partition function including the Gaussian thermal fluctuations and using the energy spectrum $E = A/8\pi$, it comes out to be

$$Z \approx \frac{1}{4\pi} e^{S_{MC}(\bar{A}) - \beta E(\bar{A})} \int_0^\infty e^{\frac{3}{4A^2}a^2}\, da \tag{IV.7}$$

where $\bar{A}$ is the horizon area at equilibrium(saddle point) and a is the fluctuation variable. The partition function is clearly undefined due to the infinite integral. Also, if one calculates the canonical entropy(S_C) taking the Gaussian thermal fluctuations into account[12], it comes out to be

$$S_C = S_{MC} - \frac{1}{2}\log \Delta$$

where

$$\Delta = \frac{K}{\partial E/\partial A}\left[\frac{\partial^2 E}{\partial A^2}\frac{\partial S_{MC}}{\partial A} - \frac{\partial^2 S_{MC}}{\partial A^2}\frac{\partial E}{\partial A}\right] \tag{IV.8}$$

evaluated at the saddle point(equilibrium configuration), K being an irrelevant positive constant. For the canonical entropy to be well defined we must have $\Delta > 0$. But, using the energy spectrum given by (III.3) it is straightforward to show that $\Delta < 0$. Thus the canonical entropy can not be defined for a QIH having an energy spectrum *linear in area*. (N does not play any role as it is kept fixed in canonical ensemble.)

In this Gaussian approximation method, the inverse temperature of the QIH at equilibrium is given by $\beta = \frac{\partial S_{MC}/\partial A}{\partial E/\partial A}$. Using (III.3) and (II.7) in the expression for β, it is easy to find that

$$\beta = 2\pi\left(1 - \frac{6}{A}\right) \tag{IV.9}$$

Comparing (IV.2) and (IV.9) one can see that the equilibrium temperature comes out to be the same in both the approaches. This is a consistency check.

V. DISCUSSION

Let us conclude with a brief summary of the crucial aspects of this paper. First of all, the *microcanonical* entropy has been derived for a generic QIH in the usual statistical mechanical approach making use of the quantum geometric framework of QIH [1, 2] and the microstates of the $SU(2)$ CS theory [3], *without* referring to any particular model of the QIH (e.g. assumption of a particular energy spectrum [5], etc.). The leading area dependent term of the *microcanonical* entropy comes out to be the usual Bekenstein-Hawking area law for fixed value(s) of BI parameter (γ) [8] for a given $\mathscr{A}/N$. A new quantum correction $N\sigma(\gamma)$ appears in the *microcanonical* entropy which is completely independent of the variable A. Of course these calculations are valid for QIH with large area (A) and large number of punctures (N) so that their small fluctuations ($\delta A \ll A, \delta N \ll N$) can be considered to be independent .

The key result of this paper is the local thermodynamic *instability* of a QIH having energy spectrum given by the equation (III.3). The result has been derived in two different approaches shown in the subsections (IV A) and (IV B). The aim was to show the role of thermal fluctuations behind the thermodynamic instability of this particular model of the QIH. In fact, though the quantum statistical analysis gives a negative specific heat, the physical picture becomes much clearer in the *thermal holographic* analysis [11, 17] where the Gaussian thermal fluctuations manifestly control the convergence criterion of the partition function. In this approach we can actually see that any arbitrary QIH is *not* thermodynamically unstable, but only those which fail to satisfy the convergence condition for the partition function. The particular QIH with energy spectrum $E = A/8\pi\ell_p$ considered in this paper is only one such example.

The two crucial outcomes of this paper, though closely resemble the results obtained in [5], are completely different and original by their own virtue. Firstly, it is shown here that the new quantum correction to the *microcanonical* entropy $N\sigma(\gamma)$ is true for any arbitrary QIH, unlike [5] where the result is true only for a QIH with a specific energy spectrum and *at a particular fixed temperature only*. From the basic knowledge of statistical mechanics (any standard text book e.g. [10]) we know that the calculation of the *microcanonical* entropy is only a combinatorial problem and does not require any issue of temperature. To calculate the *microcanonical* entropy, all one needs to know is the number of microstates subject to some *preassigned* values of the macroscopic variables defining the *microcanonical* ensemble, temperature being only a *derived* quantity. Last but not the least, in this paper the BI parameter(γ) can have one or more (to be reported in [8]) fixed values for a given $\mathscr{A}/N$ whereas γ remains a free parameter in [5].

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